ON SECOND-ORDER DIFFERENTIAL EQUATIONS WHOSE SOLUTIONS ARE BOUNDED BY A GIVEN QUANTITY

(O DIFFERENTSIAL'NYKH URAVNENIIAKH VTOROGO PORIADKA S RESHENIIAMI, OGRANICHENNYMI ZADANNOI VELICHINOI)

PMM Vol.25, No.3, 1961, pp. 583-586

A. B. SAVVIN
(Moscow)

(Received November 23, 1960)

Consider the second-order differential equation

$$\ddot{x} = f(x,\ \dot{x}) \tag{0.1}$$

where the function $f(x,\ \dot{x})$ is supposed to be continuous for $x < B$. The problem in question consists in the determination of sufficient conditions to be imposed on the right-hand side of Equation (0.1) in order that every continuously differentiable solution of this equation with initial value x_0 such that

$$x_0 < B, \quad -\infty < \dot{x}_0 < \infty \quad \text{for} \quad t = t_0 \tag{0.2}$$

remains bounded by the quantity B, i.e.

$$x < B \quad \text{for} \quad t_0 \leqslant t < \infty \tag{0.3}$$

If the initial conditions are of such a nature that the speed $\dot{x}$ is large, while the difference $B - x$ is small, then in order to insure the boundedness one requires large (but finite) accelerations which are oppositely directed to the initial velocity. Hence, even in the absence of powerful sources of energy in a system which is governed by Equation (0.1), the required accelerations may still be produced by means of a suitable arrangement of regulators. A typical example of these are hydraulic brakes. In the case of brakes, these systems may be approximately described by means of equations with singularities [1]. Sharp brakes of the impact type may not, however, produce favorable effects on the system. The properties of the equation which are analysed below allow for the choice of the regulatory system in such a way that the braking begins gradually and proceeds smoothly, while the regulated quantity does not surpass the preassigned limit.

Consider the phase space $x\dot{x}$. For $t > t_0$ the representative point may fall on the line $x = B$ only in the upper half-plane. If the solutions of Equation (0.1) with initial conditions lying in the domain D

$$B - \varepsilon < x_0 < B, \qquad 0 < \dot{x} < \infty \qquad (\varepsilon > 0) \tag{0.4}$$

are bounded by the number B, then all solutions of Equation (0.1) for arbitrary initial conditions (0.2) are also bounded.

Let the initial conditions $(x_0, \dot{x}_0) \in D$ be given. In view of the continuity of the function $f(x, \dot{x})$, the solutions of Equation (0.1) may be extended as far as the boundary of the domain D [2]. The portions of the trajectories of Equation (0.1) lying in D may be of the following types.

1) It happens that $\dot{x} \to \infty$ as $x \to x^* \leqslant B$. These solutions are defined only on a finite interval $t_0 \leqslant t < T$ and are not continuously differentiable at $t = T$. These solutions, although bounded, will not be considered here.

2) The trajectory crosses the straight line $x = B$ for $\dot{x} > 0$. The corresponding solutions are not bounded above by the number B.

3) The trajectory crosses the straight line $x = B$ for $\dot{x} = 0$. The corresponding solutions may or may not be bounded (this situation is considered in item 4 below).

4) The trajectory crosses the axis of abscissas for $x < B$. If this happens for an arbitrary initial point of the domain D, then the solutions of Equation (0.1) never exceed the number B. Below, we determine sufficient conditions for this to occur.

1. Consider the equation of phase trajectories of the form

$$\frac{d\dot{x}}{dx} = \frac{f(x, \dot{x})}{\dot{x}} \equiv g(x, \dot{x}) \tag{1.1}$$

where the function $g(x, \dot{x})$ is continuous on the domain D. If, further, this equation satisfies the conditions for the existence of a solution at a boundary point ($x_0 = B$, $\dot{x}_0 > 0$), then through this point passes an integral curve which is defined on a certain interval to the left of the straight line $x = B$ [3].

Thus, the required boundedness condition consists in the non-satisfaction, for $x = B$, $\dot{x} > 0$, of the hypotheses of all existence theorems. It is easy to prove the following non-existence theorem, which furnishes a sufficient condition for boundedness.

Theorem 1. Suppose that the right-hand side of Equation (1.1) is continuous for $x < B$. If there exists a function $m(x)$ such that

$$\lim_{x\to B} m(x) = \infty, \qquad \int_{B-\eta}^{B} m(x)\,dx = \infty \quad \text{for } \eta > 0 \tag{1.2}$$

$$|g(x, \dot{x})| \geqslant m(x) \quad \text{for } \dot{x} \geqslant 0 \tag{1.3}$$

then there does not exist any continuous integral curve of Equation (1.1) passing through the point $(B, \dot{x}_0 \geqslant 0)$, and which is defined for $x < B$.

If $f(x, \dot{x}) > 0$, then the phase trajectories approach infinity, because in this case we obtain the following necessary condition for boundedness:

$$f(x, \dot{x}) \to -\infty \quad \text{for } x \to B \qquad (\dot{x} > 0)$$

On the other hand, the second derivative $\ddot{x}$ is bounded along any trajectory, because $x < B$.

If the condition (1.3) of the theorem holds only for $\dot{x} > 0$, then the phase trajectories do not intersect the straight line $x = B$ for $\dot{x} > 0$, but may intersect it for $\dot{x} = 0$. The question of the boundedness of the solutions in this case requires further considerations (see item 4).

2. Let us introduce, in the equation of the phase trajectories, $\dot{x}$ as the independent variable and x as the dependent variable. Then we obtain the equation

$$\frac{dx}{d\dot{x}} = \frac{1}{g(x, \dot{x})} \tag{2.1}$$

This equation may have the trivial solution $x = B$; the remaining integral curves of Equations (1.1) and (2.1) coincide. If the solution $x = B$, for $\dot{x} \geqslant 0$, of (2.1) is unique, then the integral curves which pass through points $(x_0 < B, \dot{x}_0 \geqslant 0)$ cannot intersect the solution $x = B$, and hence must be bounded. This proves the following theorem.

Theorem 2. If Equation (2.1) possesses the unique solution $x = B$, for $\dot{x} \geqslant 0$, then the solutions of Equation (0.1) are bounded by B.

It should be noticed that many of the known sufficient conditions for uniqueness are too strong for the problem under consideration, because we do not need the uniqueness of every solution, but only the uniqueness of the trivial solution. For example, from the method of proof of existence and uniqueness based on the Lipschitz condition [4], it follows that the trivial solution of (2.1) will be unique provided that

$$\left|\frac{1}{g(x, \dot{x})}\right| < L\,|x - B| \tag{2.2}$$

3. Many uniqueness conditions are based on a comparison of the given equation with other equations which satisfy certain prescribed conditions. Let us employ this method of comparison in order to solve the boundedness problem.

Theorem 3. If there exists a family of continuously differentiable curves, for $\dot{x} > 0$

$$\dot{x} = \Phi(x, C) \tag{3.1}$$

which fill up completely the domain D and do not intersect the half-line $L(x = B,\ \dot{x} \geqslant 0)$, and

$$d\Phi / dx - g(x, \Phi(x, C)) > 0 \tag{3.2}$$

then the integral curves of the equation

$$d\dot{x} / dx = g(x, \dot{x}) \tag{3.3}$$

do not intersect the half-line L.

Indeed, the function $g(x,\ \dot{x})$ is continuous in the domain D. Then, the conclusion of Chaplygin's theorem on differential inequalities [5] is applicable, i.e. a solution of Equation (3.3) in the domain D lies strictly below the corresponding solution of Equation (3.1) having the same initial conditions. For $\dot{x} = 0$ they may intersect. Since all curves of the family (3.1) intersect the axis of abscissas for $x < B$, it follows that the solutions of Equation (3.1) cannot intersect the half-line L.

In particular, the family $\Phi(x,\ C)$ may form part of the family of integral curves of a differential equation

$$d\dot{x} / dx = F(x, \dot{x}), \qquad F(x, \dot{x}) > g(x, \dot{x}) \tag{3.4}$$

Let us consider the special case when

$$F(x, \dot{x}) = -F_2(\dot{x}) / F_1(x) \tag{3.5}$$

where $F_1(x)$, $F_2(\dot{x})$ are continuous functions, and

$$F_1(x) > 0 \quad \text{for } x < B, \qquad F_1(x) \geqslant 0 \quad \text{for } x = B$$
$$F_2(\dot{x}) > 0 \quad \text{for } \dot{x} > 0, \qquad F_2(\dot{x}) \geqslant 0 \quad \text{for } \dot{x} = 0$$

The equation of the family of curves is

$$\int_{\dot{x}_0}^{\dot{x}} \frac{d\dot{x}}{F_2(\dot{x})} + \int_{x_0}^{x} \frac{dx}{F_1(x)} = C \tag{3.6}$$

The integral curves will not intersect the half-line L if

$$\int_{B-\varepsilon}^{B} \frac{dx}{F_1(x)} = \infty, \qquad \int_{0}^{\eta} \frac{d\dot{x}}{F_2(\dot{x})} < \infty \qquad (\varepsilon > 0, \ \eta > 0) \tag{3.7}$$

The sufficient conditions obtained are quite general; for example, Theorem 1 follows from the comparison theorem upon choosing

$$m(x) = \frac{1}{F_1(x)}, \qquad F_2(\dot{x}) = 1$$

4. Let us now suppose that the integral curves intersect the straight line $x = B$. If the intersection is arbitrary for $\dot{x} > 0$, then the solution $x(t)$ will attain the value $x = B$ at some finite instant of time T, which contradicts the requirement (0.3). Further, the integral curve cannot be prolonged continuously for $t > T$ while at the same time remaining bounded by B.

Suppose now that an integral curve of Equation (1.1), $\dot{x} = \phi(x)$, intersects the straight line $x = B$ for $\dot{x} = 0$, that the curve passes through the point $(B, 0)$ and that it possesses a well-defined tangent there, i.e. that the following limit exists (finite or infinite):

$$\varphi'(B) = \lim_{x \to B} \frac{\varphi(x)}{x - B} \tag{4.1}$$

If this derivative is finite, i.e. the integral curve issues from the point $(B, 0)$ at an angle which differs from a right angle, then the time spent by the representative point in reaching the point $(B, 0)$ is infinite:

$$T = \int_{x_0}^{B} \frac{dx}{\varphi(x)} \to \infty \tag{4.2}$$

Thus, the required condition (0.3) holds. The second derivative $\ddot{x} = \dot{x}d\phi/dx$ along the integral curves is also bounded and approaches zero as $x = B$.

Let us now prove the following comparison theorem.

Theorem 4. Suppose that in the domain D

$$f(x, \dot{x}) < \dot{x}F(x, \dot{x}) \tag{4.3}$$

where the function $F(x, \dot{x})$ is continuous and negative in D, and the solutions of the equation

$$\ddot{x} = \dot{x}F(x, \dot{x}) \tag{4.4}$$

are bounded, for arbitrary initial conditions $(x_0, \dot{x}_0) \in D$,

$$x_F(t) < B \quad \text{for } t_0 < t < \infty \tag{4.5}$$

Then the solutions of Equation (0.1) have the same properties.

Indeed, the integral curves used for comparison are monotone, since $F(x, \dot{x})$ is negative. Since the solutions of Equation (4.4) are bounded, the integral curves of this equation either intersect the axis of abscissas for $x < B$ or they pass through the point $(B, 0)$ at an acute angle. According to Chaplygin's theorem, the integral curves of Equation (0.1) in the domain D lie strictly below the corresponding integral curves of comparison. Hence, the representative point, which moves on an integral curve of Equation (0.1), cannot reach the straight line $x = B$ in a finite time. Consequently, all solutions of Equation (0.1) are bounded.

The comparison theorem can be used very conveniently when the right-hand side is the quotient of two functions, as in (3.5). In addition to Equation (3.7), introduced in the last section, let us consider other conditions which guarantee that the solutions of (0.1) are bounded.

If

$$\int_{B-\varepsilon}^{B} \frac{dx}{F_1(x)} = \infty, \qquad \int_{0}^{\eta} \frac{d\dot{x}}{F_2(\dot{x})} = \infty \qquad (\varepsilon > 0, \eta > 0) \tag{4.6}$$

then the integral curves of Equation (4.4) pass through the point $(B, 0)$ without intersecting the straight line $x = B$ for $\dot{x} > 0$. In order that (0.3) be satisfied, the integral curves must pass the point $(B, 0)$ at an angle different from a right angle.

Filippov [6] gave sufficient conditions in order that an equation of the form $dy/dx = g(y)/h(x)$ has no solution for which $y(0) = 0$, $y'(0) = 0$, $y(x) \not\equiv 0$. Employing his results, it is easy to determine under what circumstances Equations (3.6), (3.5), subject to the conditions (4.6), have no solutions for which $\dot{x}(0) = 0$ and $d\dot{x}/dx = \infty$ for $x = 0$.

In addition to equations whose variables are separable, one may use as comparison equations other types of equations with bounded solutions.

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Translated by J.B.D.